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Hamiltonisation of classical non-holonomic systems

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Abstract. A Hamiltonisation for non-holonomic dynamical systems is developed. An example is given.

1. Introduction

Non-holonomic systems (Neimark and Fufaev 1972, Saletan and Cromer 1970, 1971) are seen to be very interesting and intriguing mechanical systems mainly when one looks for a quantisation procedure (Eden 1951, Gomes and Lobo 1979, Abud Filho *et al* 1983). The basic problem rests in the difficulty of presenting a Hamiltonian function for such systems. In fact, even the determination of a Lagrangian function describing completely the dynamics of the system is not an easy task. As an additional difficulty this Lagrangian is a singular function in the Dirac sense (Galvão and Negri 1983).

The main goal of this paper is to show how one can find a Hamiltonian function for a given non-holonomic system without using Dirac's theory (Dirac 1950, 1964). As our technique will lead to a family of Hamiltonian functions separate from subsidiary conditions it could also be expected to be an easier quantisation procedure.

To formally set up the problem which we will be interested in, let us first consider the usual Lagrangian description for a non-holonomic system. We have a free Lagrangian

$$L(q, \dot{q}, t) \equiv L(q_1, \ldots, q_N; \dot{q}_1, \ldots, \dot{q}_N, t) \tag{1.1}$$

and some subsidiary non-integrable conditions

$$\Phi_\mu(q, \dot{q}, t) = 0 \qquad \mu = 1, \ldots, K(K < N). \tag{1.2}$$

The dynamical evolution of the system in configuration space is obtained from (1.2) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \equiv \Lambda_i(q, \dot{q}, \ddot{q}, t) = \lambda^\mu \frac{\partial \Phi_\mu}{\partial \dot{q}^i} \tag{1.3}$$

where λ^μ are the Lagrange multipliers (Saletan and Cromer 1970, 1971). The summation convention is adopted and $i, j, k, \ldots = 1, \ldots, N$; $\mu = 1, \ldots, K(K < N)$.

The allowed orbits for the system are obtained by first eliminating the λ among equations (1.3) and

$$\mathrm{d}\Phi_\mu/\mathrm{d}t = 0 \tag{1.4}$$

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(which result from the imposition of the temporal preservation of the constraints), and solving a set of equations which comes to be of the following form:

$$\ddot{q}_i = f_i(q, \dot{q}, t) \tag{1.5}$$

$$\Phi_\mu(q, \dot{q}, t) = 0. \tag{1.2}$$

It is now possible to define our problem: to find a class of Hamiltonian functions that leads to the same orbits as (1.2) and (1.5) when we change from a phase space to a configuration space description. Let $H(q, p, t)$ be a member of the desired class of Hamiltonians. The corresponding canonical equations are

$$\dot{q}_i = \partial H/\partial p_i \tag{1.6}$$

$$\dot{p}_i = -\partial H/\partial q_i. \tag{1.7}$$

These equations furnish the orbits of the system in phase space. The reduction to configuration space is attained after eliminating p_i and $\dot{p}_i$ between equations (1.6) and (1.7). In fact, under suitable conditions we may differentiate (1.6) with respect to time and then eliminate p_i and $\dot{p}_i$ with the use of (1.6) and (1.7), arriving at a system of equations of the form

$$\ddot{q}_i = F_i(q, \dot{q}, t). \tag{1.5'}$$

Our problem is then to find a class of Hamiltonian functions such that solutions $q(t)$ obtained from the set (1.5′) are the same as those that come from (1.2) and (1.5). All the Hamiltonian functions of this class are able to describe equivalently (Espindola *et al* 1986) the given non-holonomic system and, in this sense, following the usual nomenclature (Hojman and Harleston 1981) they will be called *s*-equivalent Hamiltonians.

In § 2 we describe our method; in § 3 an example is given.

2. The Hamiltonisation procedure

Assume a non-holonomic system with a Lagrangian description (equations (1.1)-(1.5)). Denoting by

$$H(q, p, t)$$

a member of the desired class of *s*-equivalent Hamiltonians, we define

$$\dot{q}_i = \partial H/\partial p_i \tag{2.1}$$

and use these definitions to translate the dynamical description from configuration space to phase space by writing

$$H \equiv p_i\dot{q}_i - L(q, \dot{q}, t) = p_i\partial H/\partial p_i - L(q, \partial H/\partial p, t) \tag{2.2}$$

$$\Phi_\mu(q, \dot{q}) = 0 = \Phi_\mu(q, \partial H/\partial p). \tag{2.3}$$

Equations (2.2) and (2.3) are first-order partial differential equations for the unknown function $H(q, p, t)$. Equation (2.2) has the solution (in fact, a complete solution)

$$H = A_i p_i - L(q, A, t) \tag{2.4}$$

where A_i are, until now, arbitrary functions of the variables q_i.

The corresponding Hamilton equations (2.1) are

$$\dot{q}_i = A_i. \tag{2.5}$$

The constraints (1.2) are now

$$\Phi_\mu(q, A, t) = 0. \tag{2.6}$$

Furthermore, from (1.5) we have

$$\ddot{q}_i = \mathrm{d}A_i/\mathrm{d}t = (\partial A_i/\partial q_j)A_j + \partial A_i/\partial t = f_i(q, A, t). \tag{2.7}$$

The unknown function A_i are determined by solving the system (2.6) and (2.7). (It must be noted that the phase space constraint (2.6) works advantageously, as holonomic constraints rather than its configuration space version (1.2) that are non-holonomic.) This is a system of first-order partial differential equations for the unknown functions A_i, which in general is easy to solve (Courant and Hilbert 1962).

The second set of Hamilton equations is

$$\dot{p}_i = -\left[p_j - \frac{\partial L}{\partial A_j}\right]\frac{\partial A_j}{\partial q_i} + \frac{\partial L}{\partial q_i}. \tag{2.8}$$

As a first remark about our procedure we mention that elimination of the p_i and $\dot{p}_i$ are no longer required to go back to configuration space as it was previously pointed out when s-equivalent Hamiltonians are defined. Now we only need to use half of the Hamiltonian equations, namely (2.5). Actually this was done when equations (2.7) were written. In this sense our method can be viewed geometrically as a particular construction of the orbits in phase space in such a manner that, when projecting it in configuration space, we have the usually accepted orbits for the given non-holonomic system. We observe that we have not worried about defining the momenta p_i.

In fact, the usual definitions

$$p_i = \partial L/\partial \dot{q}_i \tag{2.9}$$

are not suitable for non-holonomic systems due to the lack of a correct Lagrangian for describing the system. This is the reason why we do not have

$$\dot{p}_i = \partial L/\partial q_i \tag{2.10}$$

in equations (2.8). Equations (2.8) define the momenta whenever necessary. For our present purpose there is no need to consider these equations.

As a second remark we mention that, after solving the system for the functions A_i, a knowledge of the constants of motion will be arrived at—a welcome additional result. Examples of this feature will be given in the next section.

3. Examples

As an example let us consider the following non-holonomic system (Gantmacher 1970)

$$L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \dot{x}^2 + \dot{y}^2 + \tfrac{1}{4}\lambda\dot{\theta}^2 - 2gy \tag{3.1}$$

(λ and g are constants),

$$\Phi = \dot{x}\sin\theta - \dot{y}\cos\theta = 0. \tag{3.2}$$

The orbits in the configuration space are given as the solution of the system

$$\begin{aligned}
&\ddot{x}=-[(\dot{x}\dot{\theta}+g)\sin\theta\cos\theta+\dot{y}\dot{\theta}\sin^2\theta]\\
&\ddot{y}=(\dot{x}\dot{\theta}+g)\cos^2\theta+\dot{y}\dot{\theta}\sin\theta\cos\theta-g\\
&\ddot{\theta}=0\\
&\Phi=0.
\end{aligned}\tag{3.3}$$

It is straightforward to verify that

$$\begin{aligned}
&x=(g\theta+g\sin\theta\cos\theta-2B\sin\theta)/2A^2+D\\
&y=(-g\cos^2\theta+2B\cos\theta)/2A^2+F\\
&\theta=At+C
\end{aligned}\tag{3.4}$$

(where A, B, C, D and F are constants),

$$H=(p_x+p_y\tan\theta)A_x+A_\theta p_\theta-A_x^2\sec^2\theta-\tfrac{1}{4}\lambda A_\theta^2+2gy\tag{3.5}$$

and (cf equations (2.7)) A_x, a_θ are obtained from

$$\begin{aligned}
&A_x\frac{\partial A_x}{\partial x}+A_x\frac{\partial A_x}{\partial y}\tan\theta+A_\theta\frac{\partial A_x}{\partial\theta}=-A_xA_\theta\tan\theta-g\sin\theta\cos\theta\\
&A_x\frac{\partial A_\theta}{\partial x}+A_x\frac{\partial A_\theta}{\partial y}\tan\theta+A_\theta\frac{\partial A_\theta}{\partial\theta}=0.
\end{aligned}$$

Solving this system we obtain

$$\begin{aligned}
&A_\theta=C_1\\
&g\cos\theta-A_xA_\theta\sec\theta=C_2\\
&yA_\theta^2+A_\theta A_x-\tfrac{1}{2}g\cos^2\theta-g/4=C_3\\
&xA_\theta^2-A_\theta A_x\tan\theta+g[(\sin 2\theta)/2-\theta]/2=C_4
\end{aligned}$$

where C_1, C_2, C_3 and C_4 are constants. Hence, by writing

$$\begin{aligned}
&C_1=F(C_3,C_4)\\
&C_3=G(C_3,C_4)
\end{aligned}$$

with F and G arbitrary functions, we may write

$$A_\theta=F(yA_\theta^2+A_\theta A_x-(g/2)\cos^2\theta-g/4,\ xA_\theta^2-A_\theta A_x\tan\theta+(g/4)\sin 2\theta-g\theta/2)\tag{3.6}$$

$$\begin{aligned}
g\cos\theta-A_\theta A_x\sec\theta=G(&yA_\theta^2+A_\theta A_x-(g/2)\cos^2\theta-g/4,\ xA_\theta^2\\
&-A_\theta A_x\tan\theta+(g/4)\sin 2\theta-g\theta/2).
\end{aligned}\tag{3.7}$$

A particular solution can be selected:

$$\begin{aligned}
&A_\theta=\text{constant}=A\\
&A_x=(g\cos^2\theta-B\cos\theta)/A\qquad B=\text{constant}.
\end{aligned}$$

With this solution we have

$$\begin{aligned}
H=(p_x+p_y\tan\theta)(g\cos^2\theta-B\cos\theta)/A+Ap_\theta&\\
-(g\cos^2\theta-B\cos\theta)^2(\sec^2\theta)/A^2-\lambda A^2/4+2gy.&
\end{aligned}$$

The canonical Hamiltonian equations can now be given explicitly. Half of them are

$$\dot{x} = (g\cos^2\theta - B\cos\theta)/A$$
$$\dot{y} = (g\sin\theta\cos\theta - B\sin\theta)/A$$
$$\dot{\theta} = A$$

and it is easily seen after integration that we have the solutions (3.4).

As a final remark we observe, as before, that we have also obtained constants of motion for the non-holonomic system. In the present case they are

$$F_1 = \dot{\theta}$$
$$F_2 = g\cos\theta - \dot{\theta}\dot{x}\sec\theta$$
$$F_3 = y\dot{\theta}^2 + \dot{\theta}\dot{x} - g(\cos^2\theta)/2 - g/4$$
$$F_4 = x\dot{\theta}^2 - \dot{\theta}\dot{x}\tan\theta + (g/4)\sin 2\theta - g\theta/2.$$

4. Final remarks

Our main result (2.4) exhibits the Hamiltonian, for a non-holonomic system, as a linear function of the momenta p_i, i.e., a singular system (in Dirac's nomenclature), as it must be, taking into account a previous result (Galvão and Negri 1983).

Another family of equivalent Hamiltonians is given by

$$\tilde{H}(Q, P) = P_i A_i(Q) + G(Q)$$

where G is an arbitrary function (affecting only the definition of the momenta p_i). If the functions A_i are the ones previously defined, this Hamiltonian will lead to the same orbits in configuration space. In this sense we could say that this $\tilde{H}$ and the Hamiltonian previously obtained are related by some 'gauge transformation'. We shall analyse transformation properties of this theory in a forthcoming paper.

References

Abud F M, Gomes L C, Simão F R A and Coutinho F A B 1983 *Rev. Bras. Fís.* **13** 384

Courant R and Hilbert D 1962 *Methods of Mathematical Physics* vol 2 (New York: Interscience) ch 2, p 140

Dirac P A M 1950 *Can. J. Math.* **2** 129

—— 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University)

Eden R J 1951 *Proc. R. Soc.* A **205** 564, 583

Espindola O, Teixeira N L and Espindola M L 1986 *J. Math. Phys.* **27** 151

Galvão C A P and Negri L J 1983 *J. Phys. A: Math. Gen.* **16** 4183

Gantmacher F 1970 *Lectures in Analytical Mechanics* (Moscow: Mir) 1st edn, ch 1, p 63

Gomes L C and Lobo R 1979 *Rev. Bras. Fís.* **9** 459

Hojman S and Harleston H 1981 *J. Math. Phys.* **22** 1414

Neimark Ju I and Fufaev N A 1972 *American Mathematical Society Translations Math. Monographs* vol 33 (Providence, RI: Am. Math. Soc.) ch 1, 3

Saletan E J and Cromer A H 1970 *Am. J. Phys.* **38** 892

—— 1971 *Theoretical Mechanics* (New York: Wiley) ch II, IV, § 5e